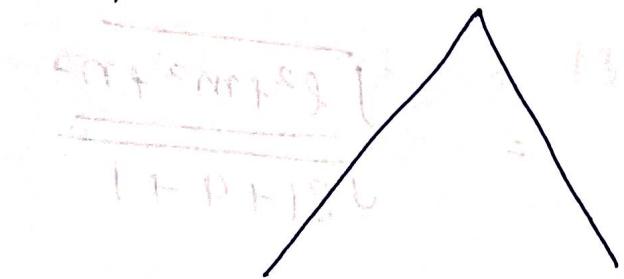


* Linear Differential equation with variable coefficients



Cauchy's linear
equation

Legendre's linear
equation

→ Cauchy's linear equation is

$$(p_0 x^n D^n + p_1 x^{n-1} D^{n-1} + \dots + p_{n-1} x D + p_n) y = f(x)$$

where $D = \frac{d}{dx}$

To Reduce the linear equation with constant coeffs.

(i) put $x = e^z$ so that $z = \log x$ and $\frac{d}{dz} = \theta$

(ii) Replace $xD \rightarrow \theta$

$$x^2 D^2 \rightarrow \theta(\theta-1)$$

$$\theta = p\theta + p'x + \frac{p''x^2}{2!} = (p)_2$$

$$x^n D^n \rightarrow \theta(\theta-1)(\theta-2)\dots(\theta-n+1)$$

$$\theta = p\theta + (p')_2 x + (p'')_3 \frac{x^2}{2!} + \dots = (p)_n$$

(iii) equation reduces to linear equation with constant coefficients

$$y\theta + \left[p\theta + \frac{p'b}{x} x \right] - \left[p'x^2 + \frac{p''x^3}{2!} \right] = b$$

(iv) solve it for y in terms of z and then put

$$y\theta + p\theta - z = \log x \quad y\theta + \frac{pb}{x} x + \frac{p'b}{x^2} x^2 + \frac{p''b}{x^3} x^3$$

\Rightarrow Reduces to Differential equation :-

$$[P_0(ax+b)^n \theta^n + P_1(ax+b)^{n-1} \theta^{n-1} + \dots + P_{n-1}(ax+b)\theta + P_n]y = f(x)$$

$$\text{where } \theta = \frac{d}{dx} + \frac{b}{ax+b} x - \frac{p''b^2 x}{(ax+b)}$$

(i) Put $ax+b = e^z$ so that $y(z = \log(ax+b))$ and $\frac{d}{dz} = \theta$

(ii) Replace $(ax+b)\theta = a\theta = \frac{b}{ab} (e^z - 1) = \theta - 1$

$$(ax+b)^2 \theta = a^2 \theta (\theta-1)$$

(iii) equation reduces to linear equation with constant coeff.

(iv) solve it for y in terms of z , then put $z = \log x$.

Ex-1 find the integrating factor of the equation

$$l_2(y) = x^2y'' + 6xy' + 6y = 0 \quad \dots \textcircled{1}$$

and hence solve it.

→ Here $\bar{l}_2(y) = (-1)^2 \frac{d^2}{dx^2}(x^2y) + (-1)^1 \frac{d}{dx}(6xy) + 6y = 0$

$$\frac{d}{dx} \left[x^2 \frac{dy}{dx} + 2xy \right] - \left[6x \frac{dy}{dx} + 6y \right] + 6y = 0$$

$$x^2 \frac{d^2y}{dx^2} + 2x \frac{dy}{dx} + 2x \frac{dy}{dx} + 2y - 6x \frac{dy}{dx} - 6y + 6y = 0$$

$$x^2 \frac{d^2y}{dx^2} + 4x \frac{dy}{dx} + 2y - 6x \frac{dy}{dx} = 0$$

$$x^2 \frac{d^2y}{dx^2} - 2x \frac{dy}{dx} + 2y = 0 \quad \textcircled{2}$$

$$(x^2 D^2 - 2x D + 2)y = 0 \quad \text{where } D = \frac{d}{dx}$$

$$\text{Put } x = e^z, \frac{d}{dz} = \theta D = D(D+xD)$$

$$(\theta(\theta-1) - 2\theta + 2)y = 0$$

Auxiliary equation is

$$\theta^2 - 3\theta + 2 = 0$$

$$\theta^2 - 2\theta - \theta + 2 = 0 \quad \text{for I.P. term}$$

$$\theta(\theta - 2) - 1(\theta - 2) = 0$$

$$(\theta - 2)(\theta - 1) = 0$$

$$\theta = 1, 2$$

\therefore Eq. ① is complete. But

complete solution of ② is

$$y = c_1 e^x + c_2 e^{2x} \quad \text{by method}$$

$$y = c_1 x + c_2 x^2$$

\therefore two L.I. solns of eqn ② are x and x^2

\therefore Two I.F. of given eqn ① are x and x^2

Multiply eqn ① by I.F. x

$$x^3 y'' + 6x^2 y' + 6xy = 0 \quad \text{--- } ③$$

If must be exact

$$\begin{array}{c|ccccc}
 & x^3 & & 6x^2 & & 6x \\
 -1) & & & -3x^2 & & -6x \\
 \hline
 -1) & x^3 & & 3x^2 & & 0 \\
 & & & -3x^2 & & \\
 \hline
 & x^3 & & 0 & &
 \end{array}$$

\therefore Integral eqn of ③ is $x^3 y + 3x^2 y = C_1$

$$x^3 y' + 3x^2 y = C_1 \quad \text{--- (4)}$$

and Integral eqn of ④ is

$$x^3 y = C_1 x + C_2$$

where C_1 and C_2 are constants is the solution of ①.