* Linear Differential equation with variable coefficients

Cauchy's linear equation

Legendre's linear equation

⇒ Cauchy's linear equation is

$$(p_0 x^n D^n + p_1 x^{n-1} D^{n-1} + \cdots + p_{n-1} x D + p_n) y = f(x)$$

where $D = \frac{d}{dx}$

To Reduce the linear equation with constant coeffs.

(i) put $x = e^z$ so that $z = \log x$ and $\frac{d}{dz} = \theta$

(ii) Replace $xD \rightarrow \theta$

$$x^2D^2 \rightarrow \theta(\theta-1)$$

$$\dots \dots$$

$$x^nD^n \rightarrow \theta(\theta-1)(\theta-2)\dots(\theta-n+1)$$

(iii) equation reduces to linear equation with constant coefficients

(iv) Solve it for y in terms of z and then put $z = \log x$.

⇒ <u>Legendre's Differential equation</u> :-

$$[P_0(ax+b)^n D^n + P_1(ax+b)^{n-1}D^{n-1} + \dots + P_{n-1}(ax+b)D + P_n]y = f(x)$$

where $D = \frac{d}{dx}$

(i) put $ax+b = e^z$ so that $z = \log(ax+b)$ and $\frac{d}{dz} = \theta$

(ii) Replace $(ax+b)D = a\theta$

$$(ax+b)^2D = a^2\theta(\theta-1)$$

(iii) equation reduces to linear equation with constant coeff.

(iv) Solve it for y in terms of z, then put $z = \log x$.

Ex-1 find the Integrating factor of the equation

$$L_2(y) = x^2 y'' + 6xy' + 6y = 0 \quad \text{--- (1)}$$

and hence solve it.

→ Here $\bar{L}_2(y) = (-1)^2 \frac{d^2}{dx^2}(x^2 y) + (-1)^1 \frac{d}{dx}(6xy) + 6y = 0$

$$\frac{d}{dx}\left[x^2\frac{dy}{dx} + 2xy\right] - \left[6x\frac{dy}{dx} + 6y\right] + 6y = 0$$

$$x^2\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} + 2x\frac{dy}{dx} + 2y - 6x\frac{dy}{dx} - 6y + 6y = 0$$

$$x^2\frac{d^2y}{dx^2} + 4x\frac{dy}{dx} + 2y - 6x\frac{dy}{dx} = 0$$

$$x^2\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = 0 \quad \text{--- (2)}$$

$(x^2D^2 - 2xD + 2)y = 0$ where $D = \frac{d}{dx}$

Put $x = e^z$, $\frac{d}{dz} = \theta$

$$(\theta(\theta-1) - 2\theta + 2)y = 0$$

Auxiliary equation is

$$\theta^2 - 3\theta + 2 = 0$$

$\theta^2 - 2\theta - \theta + 2 = 0$

$\theta(\theta - 2) - 1(\theta - 2) = 0$

$(\theta - 2)(\theta - 1) = 0$

$\theta = 1, 2$

compute solution of ② is

$y = c_1 e^{z} + c_2 e^{2z}$

$y = c_1 x + c_2 x^2$

$\therefore$ two L.I solns of eqn ② are x and x^2

$\therefore$ Two I.F of given eqn ① are x and x^2

Multiply eqn ① by I.F x

$x^3 y'' + 6x^2 y' + 6xy = 0$ —— ③

It must be Exact

$-1)$	x^3	$6x^2$ $-3x^2$	$6x$ $-6x$
$-1)$	x^3	$3x^2$ $-3x^2$	0
	x^3	0	

$\therefore$ Integral eqn of (3) is

$$x^3 y' + 3x^2 y = c_1 \quad \text{——} (4)$$

and Integral eqn of (4) is

$$x^3 y = c_1 x + c_2$$

where c_1 and c_2 are constants is the solution of (1).